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Cardinal Interpolation by Polynomial Splines: Interpolation of Data with Exponential Growth

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Let $m \in \mathbb{N}$ and define S_m to be the class of functions $f \in C^{m-1}(\mathbb{R})$ which, in each $[j-1, j]$ $(j \in \mathbb{Z})$, coincide with some real polynomial of degree $\leqslant m$. We study the cardinal spline interpolation problem of constructing an element $s \in S_m$ with $s(j-\lambda) = y_j$, $j \in \mathbb{Z}$, where $\lambda \in (0, 1]$ is a translation parameter. Under some natural conditions on λ and m, ter Morsche (*in* "Spline Functions" (K. Böhmer, G. Meinardus, and W. Schempp, Eds.), pp. 210–219, Springer-Verlag, Berlin/Heidelberg/New York, 1976) and Schoenberg (*J. Approx. Theory* **6** (1972), 404–420; *in* "Studies in Spline-Functions and Approximation Theory" (S. Karlin, Ch. A. Micchelli, A. Pinkus, I. J. Schoenberg, Eds.), pp. 251–276, Academic Press, New York, 1976) have proved that this problem has a unique solution of power growth, provided that the interpolation data are of power growth and that this solution can be given by a series of Lagrangian splines converging locally uniformly. In what follows we prove an analogous result for exponential growth conditions instead of power growth conditions. Moreover, we extend the concept of extremal bases, given by Reimer (in "Approximation Theory III" (E. Cheney, Ed.), pp. 723–728, Academic Press, New York, 1980), to topological bases of normed spaces with infinite dimension and apply this concept to the subspace of all bounded functions of S_m.

1. INTRODUCTION

Let $m \in \mathbb{N}$ and let S_m be the class of cardinal spline functions of degree m, consisting of all functions $s \in C^{m-1}(\mathbb{R})$ with

$$s(j-1+t) = p_j(t), \qquad p_j \in \mathbb{P}_m,$$

for $t \in [0, 1]$ and $j \in \mathbb{Z}$, where $\mathbb{P}_m$ denotes the linear space of real polynomials of degree not exceeding m.

Further, let $\lambda \in (0, 1]$ be a translation parameter and $y = (y_j)_{j \in \mathbb{Z}}$ be a prescribed sequence of real numbers. We study the problem of finding a spline $s \in S_m$ satisfying

$$s(j-\lambda) = y_j, \qquad j \in \mathbb{Z}, \tag{1}$$

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which is an element of some prescribed linear space $V \subset S_m$. In accordance to Schoenberg [14] we call this problem a cardinal interpolation problem and denote it by the symbol

$$\mathrm{CIP}(y, V).$$

In what follows we derive the interpolation theory for the space

$$S_m^\infty := \{s \in S_m : s \text{ bounded}\}$$

by a method, which can also be applied to the spaces

$$S_m^{(\beta_1,\beta_2)} := \{s \in S_m : s(x) = \mathcal{O}(\beta_1^x),\ x \to \infty,$$
$$s(x) = \mathcal{O}(\beta_2^{|x|}),\ x \to -\infty\},$$

$\beta_1, \beta_2 \in \mathbb{R}^+$, and which then provides for new results. In order to provide that the cardinal interpolation problems $\mathrm{CIP}(y, S_m)$, $\mathrm{CIP}(y, S_m^{(\beta_1,\beta_2)})$ are not unsolvable in advance, we assume that y is an element of

$$Y^\infty := \{y = (y_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} : y \text{ bounded}\},$$
$$Y^{(\beta_1,\beta_2)} := \{y \in (y_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} : y_j = \mathcal{O}(\beta_1^j),\ j \to \infty,$$
$$y_j = \mathcal{O}(\beta_2^{|j|}),\ j \to -\infty\},$$

respectively ($\beta_1, \beta_2 \in \mathbb{R}^+$).

Under the assumption

$$(\lambda = 1 \Rightarrow m \equiv 1 \bmod 2) \wedge (\lambda = \tfrac{1}{2} \Rightarrow m \equiv 0 \bmod 2) \tag{2}$$

ter Morsche [6] and Schoenberg [14, 16] have examined the problem $\mathrm{CIP}(y, V)$ for linear spaces which are given by polynomial growth conditions. They showed that $\mathrm{CIP}(y, S_m)$ has one and only one solution of power growth, if $y \in \mathbb{R}^{\mathbb{Z}}$ is of power growth, and that this solution is given by a series of Lagrangian splines converging locally uniformly.

In what follows we prove similar results for exponential growth conditions instead of polynomial growth conditions. To this end we derive a criterion for the convergence/divergence of the series of Lagrangian splines used by ter Morsche and Schoenberg, as is usual in the theory of power series. This is a generalization of the work of Reimer [9] who dealt with the case $\lambda = 1$, $m \equiv 1 \bmod 2$. Finally we deal with the case

$$(\lambda = 1 \wedge m \equiv 1 \bmod 2) \vee (\lambda = \tfrac{1}{2} \wedge m \equiv 0 \bmod 2), \tag{3}$$

which is of importance, as in this case the Lagrangians of the problem $\mathrm{CIP}(y, S_m^\infty)$ have the extremal property that their supremum-norms are given by one, see Reimer [8], Siepmann [17]. Besides, we extend the concept of extremal algebraic bases for normed spaces, given by Reimer [7], to the case of topological bases for normed spaces with infinite dimension and apply this concept to the space S_m^∞.

2. Convergence of Lagrangian Spline Series

Like Meinardus and Merz [3], Merz [4], ter Morsche [5, 6], and Reimer [8, 9] we make use of the generalized Euler–Frobenius polynomials $H_n: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, defined by

$$H_n(t, z) = (1-z)^{n+1} \cdot \left(t + z \frac{\partial}{\partial z}\right)^n \left(\frac{1}{1-z}\right), \tag{4}$$

$t, z \in \mathbb{C}$, $|z| < 1$, $n \in \mathbb{N}_0$. Due to ter Morsche [5] the zeroes $z_\nu^{(n)}(t)$ of $H_n(t, \cdot)$ are real distinct and non-positive, if $t \in [0, 1]$, $n \geqslant 1$, and may be enumerated such that

$$z_1^{(n)}(t) < z_2^{(n)}(t) < \cdots < z_n^{(n)}(t) \leqslant 0 \tag{5}$$

holds (with $z_1^{(n)}(1) = -\infty$; compare (7)). By the definition of the H_n we obtain the following properties (which are proved in ter Morsche [5] or Siepmann [17]):

$$\frac{\partial}{\partial t} H_n(t, z) = n \cdot (1-z) \cdot H_{n-1}(t, z), \qquad n \in \mathbb{N}, \tag{6}$$

$$H_n(0, z) = z \cdot H_n(1, z), \qquad n \in \mathbb{N}, \tag{7}$$

$$H_n(1-t, z) = z^n \cdot H_n(t, z^{-1}), \qquad z \neq 0, \tag{8}$$

$$z_\nu^{(n)}(t) \text{ is strictly monotonic decreasing with respect to } t \in [0, 1], \tag{9}$$

$$a_{\nu,\lambda}^{(n)}(t) := \frac{H_n(t, z_\nu^{(n)}(1-\lambda))}{H_n^z(1-\lambda, z_\nu^{(n)}(1-\lambda))} \begin{cases} <0 & \text{if } \; t \in [0, 1-\lambda) \\ =0 & \text{if } \; t = 1-\lambda, \\ >0 & \text{if } \; t \in (1-\lambda, 1) \end{cases} \tag{10}$$

$$\nu = 1, \ldots, m, \; \lambda \in (0, 1],$$

$$H_n(t, z) = n! \sum_{j=0}^{n} B_{0,n}(j+t)\, z^j, \tag{11}$$

$$t \in [0, 1], \qquad z \in \mathbb{R}, \qquad n \in \mathbb{N},$$

where $B_{0,n}$ denotes the unique element of S_n with supp $B_{0,n} = [0, n+1]$ and $\int_{\mathbb{R}} B_{0,n}(x)\, dx = 1$ (compare Meinardus [1]).

We base our ideas on the following representation of the Lagangian $l_{0,\lambda}^N$ of the problem CIP(y, S_m^N), where S_m^N is defined by

$$S_m^N := \{s \in S_m : s(x+N) = s(x),\ x \in \mathbb{R}\}.$$

Note that due to ter Morsche [5] CIP(y, S_m^N) has a unique solution for any data $y \in Y^N$,

$$Y^N := \{y = (y_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} : y_{j+N} = y_j,\ j \in \mathbb{Z}\},$$

provided that the parameters λ, m, and N satisfy the condition

$$N \equiv 1 \bmod 2$$

or

$$N \equiv 0 \bmod 2 \wedge m \equiv 0 \bmod 2 \wedge \lambda \neq 1 \tag{12}$$

or

$$N \equiv 0 \bmod 2 \wedge m \equiv 1 \bmod 2 \wedge \lambda \neq \tfrac{1}{2}.$$

Let $l_{0,\lambda}^N$ be the unique solution of CIP(y, S_m^N) for the data

$$y_k := \begin{cases} 0 & \text{if } k \not\equiv 0 \bmod N, \\ 1 & \text{if } k \equiv 0 \bmod N, \end{cases}$$

$k \in \mathbb{Z}$, and define $q_{j,\lambda}^N \in \mathbb{P}_m$ by

$$q_{j,\lambda}^N(t) = l_{0,\lambda}^N(j-1+t), \qquad t \in [0, 1],$$

$j \in \mathbb{Z}$. Using the abbreviations

$$z_{\mu,\lambda} := z_\mu^{(m)}(1-\lambda) \qquad \text{and} \qquad a_{\mu,\lambda}(t) = a_{\mu,\lambda}^{(m)}(t)$$

we have the identity

$$q_{j,\lambda}^N(t) = (1-t)^m \lambda^{-m}\, \delta_{0,j} + \sum_{\mu=1}^{m} a_{\mu,\lambda}(t)(z_{\mu,\lambda}^{N-j-1}/(1-z_{\mu,\lambda}^N)) \tag{13}$$

for $t \in [0, 1]$, $j = 0, 1, \ldots, N-1$. In case of $\lambda = 1 \wedge m \equiv 1 \bmod 2$ Reimer [8] has derived similar equations from a well-known representation by Meinardus and Merz [3] and Merz [4] by means of the residual theorem. This method has been extended to the other cases noted in (12) by Siepmann [17]. (13) may be proved in the same way, but it became possible to derive it without using the residual theorem by purely algebraic methods (Reimer and Siepmann [10]).

In order to obtain a representation of the Lagrangians of the interpolation problem CIP(y, S_m^∞) we need the following notation.

Let $t \in [0, 1)$ and $n \in \mathbb{N}\setminus\{1\}$. Define

$$
\begin{aligned}
I_1^{(n)}(t) &:= \{\mu \in \{1, ..., n\} \colon |z_\mu^{(n)}(t)| \geqslant 1\},\\
I_2^{(n)}(t) &:= \{\mu \in \{1, ..., n\} \colon |z_\mu^{(n)}(t)| \leqslant 1\},\\
r_1^{(n)}(t) &:= \max I_1^{(n)}(t),\\
r_2^{(n)}(t) &:= \min I_2^{(n)}(t),\\
\zeta_1^{(n)}(t) &:= z_{r_1^{(n)}(t)}^{(n)}(t),\\
\zeta_2^{(n)}(t) &:= z_{r_2^{(n)}(t)}^{(n)}(t).
\end{aligned}
$$

Note that $I_j^{(n)}(t) \neq \varnothing$, $j = 1, 2$, and $\zeta_1^{(n)}(t) \leqslant -1$, $-1 \leqslant \zeta_2^{(n)}(t) \leqslant 0$ hold for $t \in [0, 1)$.

By the limit process $N \to \infty$ we have

LEMMA 1. *Let* $m \in \mathbb{N}\setminus\{1\}$, $\lambda \in (0, 1]$, $v \in \{0, 1, ..., m-1\}$. *Then the sequence* $((d/dx)^v\, l_{0,\lambda}^{2N+1}(x))_{N\in\mathbb{N}}$ *converges locally uniformly on* $\mathbb{R}$. *The function* $l_{0,\lambda}^\infty \colon \mathbb{R} \to \mathbb{R}$, $l_{0,\lambda}^\infty(x) := \lim_{N\to\infty} l_{0,\lambda}^{2N+1}(x)$, *has the representation*

$$l_{0,\lambda}^\infty(j-1+t) = q_{j,\lambda}^\infty(t), \qquad t \in [0, 1], j \in \mathbb{Z},$$

with

$$
q_{j,\lambda}^\infty(t) := \begin{cases} (1-t)^m \lambda^{-m}\, \delta_{0,j} - \displaystyle\sum_{\mu \in I_1^{(m)}(1-\lambda)}^{*} a_{\mu,\lambda}^{(m)}(t) \cdot (z_\mu^{(m)}(1-\lambda))^{-j-1} \\ \qquad \text{for} \quad j \in \mathbb{N}_0, \\ \displaystyle\sum_{\mu \in I_2^{(m)}(1-\lambda)}^{*} a_{\mu,\lambda}^{(m)}(t) \cdot (z_\mu^{(m)}(1-\lambda))^{-j-1} \qquad \text{for} \quad -j \in \mathbb{N}, \end{cases} \tag{14}
$$

where the asterisk in the sums means that every term of the sum belonging to a summation index μ *with* $|z_\mu^{(m)}(1-\lambda)| = 1$ *is to be weighted by the factor* $\frac{1}{2}$.

Moreover $l_{0,\lambda}^\infty$ *is a solution of* CIP($(\delta_{0,j})_{j\in\mathbb{Z}}$, S_m^∞) *and we have the asymptotic*

$$l_{0,\lambda}^\infty(j-1+t) = \mathcal{O}(|\zeta_1^{(m)}(1-\lambda)|^{-j}), \qquad j \to \infty,$$

$$l_{0,\lambda}^\infty(j-1+t) = \mathcal{O}(|\zeta_2^{(m)}(1-\lambda)|^{-j}), \qquad j \to -\infty.$$

Proof. Note that, by (13), the limits

$$\begin{aligned} \left(\frac{d}{dt}\right)^{\nu} q_{j,\lambda}^{\infty}(t) &= \lim_{N\to\infty} \left(\frac{d}{dt}\right)^{\nu} q_{j,\lambda}^{2N+1}(t), \qquad j \in \mathbb{N}_0, \\ \left(\frac{d}{dt}\right)^{\nu} q_{j,\lambda}^{\infty}(t) &= \lim_{N\to\infty} \left(\frac{d}{dt}\right)^{\nu} q_{j+2N+1,\lambda}^{2N+1}(t), \qquad -j \in \mathbb{N}, \end{aligned} \tag{15}$$

are uniform with respect to $t \in [0, 1]$, $\nu = 0, 1, ..., m-1$. Thus the properties of $l_{0,\lambda}^{\infty}$, given in Lemma 1, are direct consequences of the interpolation properties of the regularity properties of $l_{0,\lambda}^{2N+1}$, and of (5).

Remark. If assumption (2) holds, then we even have that the limits

$$\left(\frac{d}{dx}\right)^{\nu} l_{0,\lambda}^{\infty}(x) = \lim_{N\to\infty} \left(\frac{d}{dx}\right)^{\nu} l_{0,\lambda}^{N}(x)$$

exist, $\nu = 0, 1, ..., m-1$; in this case the representation of the Lagrangian $l_{0,\lambda}^{\infty}$ coincides with that given by ter Morsche [6].

Next we define the functions $l_{j,\lambda}^{\infty} \in S_m^{\infty}$, $j \in \mathbb{Z}$, by

$$l_{j,\lambda}^{\infty}(x) := l_{0,\lambda}^{\infty}(x-j), \qquad x \in \mathbb{R},$$

and deal with the problem of convergence/divergence of the series

$$\sum_{j=-\infty}^{\infty} y_j l_{j,\lambda}^{\infty}(x), \qquad x \in \mathbb{R},$$

which is called convergent (uniformly convergent, locally uniformly convergent), if both the series

$$\sum_{j=0}^{\infty} y_j l_{j,\lambda}^{\infty}(x) \qquad \text{and} \qquad \sum_{j=-\infty}^{-1} y_j l_{j,\lambda}^{\infty}(x)$$

converge (converge uniformly, converge locally uniformly).

Further, let us introduce the abbreviations

$$\begin{gathered} z_{\mu,\lambda} := z_{\mu}^{(m)}(1-\lambda), \qquad a_{\mu,\lambda}(t) := a_{\mu,\lambda}^{(m)}(t), \qquad \mu = 1, 2, ..., m, \\ \zeta_{j,\lambda} := \zeta_j^{(m)}(1-\lambda), \qquad I_{j,\lambda} := I_j^{(m)}(1-\lambda), \qquad j = 1, 2, \\ r_{j,\lambda} := r_j^{(m)}(1-\lambda), \qquad j = 1, 2. \end{gathered}$$

As a consequence of Lemma 1 the formal series

$$s(k-1+t) := \sum_{j=-\infty}^{\infty} y_j l_{j,\lambda}^{\infty}(k-1+t)$$

satisfy the following identities for $k \in \mathbb{Z}$, $t \in [0, 1]$:

$$
\begin{aligned}
s(k-1+t) &= \sum_{j=-\infty}^{\infty} y_j l_{0,\lambda}^{\infty}(k-1+t-j) \\
&= \sum_{j=k+1}^{\infty} y_j q_{k-j,\lambda}^{\infty}(t) + \sum_{j=-k}^{\infty} y_{-j} q_{k+j,\lambda}^{\infty}(t) \\
&= \sum_{j=2}^{\infty} y_{k+j-1} q_{1-j,\lambda}^{\infty}(t) + \sum_{j=0}^{\infty} y_{k-j} q_{j,\lambda}^{\infty}(t) \\
&= y_k(1-t)^m \lambda^{-m} + \sum_{j=2}^{\infty} y_{k+j-1} \cdot \sum_{\mu \in I_{2,\lambda}}^{*} a_{\mu,\lambda}(t)\, z_{\mu,\lambda}^{j-2} \\
&\quad - \sum_{j=0}^{\infty} y_{k-j} \cdot \sum_{\mu \in I_{1,\lambda}}^{*} a_{\mu,\lambda}(t)\, z_{\mu,\lambda}^{-j-1} \\
&= y_k(1-t)^m \lambda^{-m} + \sum_{j=0}^{\infty} y_{k+j+1} \zeta_{2,\lambda}^{j} \cdot \sum_{\mu \in I_{2,\lambda}}^{*} a_{\mu,\lambda}(t) \left(\frac{z_{\mu,\lambda}}{\zeta_{2,\lambda}}\right)^j \\
&\quad - \sum_{j=1}^{\infty} y_{k-j+1} \zeta_{1,\lambda}^{-j} \cdot \sum_{\mu \in I_{1,\lambda}}^{*} a_{\mu,\lambda}(t) \left(\frac{\zeta_{1,\lambda}}{z_{\mu,\lambda}}\right)^j.
\end{aligned} \tag{16}
$$

Now define the polynomials $A_{j,\lambda}$, $B_{j,\lambda}$, $j \in \mathbb{Z}$, by

$$
A_{j,\lambda}(t) = \sum_{\mu \in I_{2,\lambda}}^{*} a_{\mu,\lambda}(t) \cdot \left(\frac{z_{\mu,\lambda}}{\zeta_{2,\lambda}}\right)^j,
$$

$$
B_{j,\lambda}(t) = \sum_{\mu \in I_{1,\lambda}}^{*} a_{\mu,\lambda}(t) \cdot \left(\frac{\zeta_{1,\lambda}}{z_{\mu,\lambda}}\right)^j,
$$

$t \in [0, 1]$. By (5) and (10) the inequalities

$$
0 < |a_{r_{2,\lambda},\lambda}(t)| \leqslant |A_{j,\lambda}(t)| \leqslant \sum_{\mu \in I_{2,\lambda}}^{*} |a_{\mu,\lambda}(t)| \leqslant A,
$$

$$
0 < |a_{r_{1,\lambda},\lambda}(t)| \leqslant |B_{j,\lambda}(t)| \leqslant \sum_{\mu \in I_{1,\lambda}}^{*} |a_{\mu,\lambda}(t)| \leqslant B,
$$

hold for $t \in [0, 1) \backslash \{1-\lambda\}$, $j \in \mathbb{Z}$ (see Reimer [9], Siepmann [17]). Hence we conclude that the identities

$$
\begin{aligned}
\limsup_{j \to \infty} \sqrt[j]{|y_j|} &= \limsup_{j \to \infty} \sqrt[j]{|y_{j+k+1} \cdot A_{j,\lambda}(t)|}, \\
\limsup_{j \to -\infty} \sqrt[|j|]{|y_j|} &= \limsup_{j \to -\infty} \sqrt[|j|]{|y_{j+k+1} \cdot B_{j,\lambda}(t)|}
\end{aligned} \tag{17}
$$

hold for $k \in \mathbb{Z}$, $t \in [0, 1) \backslash \{1-\lambda\}$. Now we can prove

THEOREM 1. *Let* $m \in \mathbb{N}\backslash\{1\}$, $\lambda \in (0, 1]$, *and* $(y_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ *and define*

$$R_+ := (\limsup_{j\to\infty} \sqrt[j]{|y_j|})^{-1}, \qquad R_- := (\limsup_{j\to-\infty} \sqrt[|j|]{|y_j|})^{-1}. \tag{18}$$

Then the series

$$s(x) := \sum_{j=-\infty}^{\infty} y_j l_{j,\lambda}^{\infty}(x) \tag{19}$$

is convergent for $x \in (1-\lambda)\cdot\mathbb{Z}$ *by definition, for* $x \in \mathbb{R}\backslash(1-\lambda)\cdot\mathbb{Z}$, *if*

$$|\zeta_2^{(m)}(1-\lambda)| < R_+ \qquad \textit{and} \qquad |\zeta_1^{(m)}(1-\lambda)|^{-1} < R_- \tag{20}$$

holds, and it is divergent for $x \in \mathbb{R}\backslash(1-\lambda)\cdot\mathbb{Z}$, *if*

$$|\zeta_2^{(m)}(1-\lambda)| > R_+ \qquad \textit{or} \qquad |\zeta_1^{(m)}(1-\lambda)|^{-1} > R_-. \tag{21}$$

Moreover, if condition (20) *holds, then the formal derivations*

$$s_\nu(x) := \sum_{j=-\infty}^{\infty} y_j \left(\frac{d}{dx}\right)^\nu l_{j,\lambda}^{\infty}(x)$$

converge locally uniformly in $\mathbb{R}$, $\nu = 0, 1, ..., m-1$, *and* s *is a solution of* $\mathrm{CIP}((y_j)_{j\in\mathbb{Z}}, S_m)$.

Proof. Convergence/divergence of the series (19) is an immediate consequence of (17). Now let (20) hold. If we substitute the polynomials $a_{\mu,\lambda}$ in (16) by $(d/dt)^\nu a_{\mu,\lambda}$, $\nu = 0, 1, ..., m-1$, then we derive the uniform convergence of $(d/dt)^\nu s(k-1+t)$ with respect to $t \in [0, 1]$ by means of the inequalities

$$\limsup_{j\to\infty} \sqrt[j]{\left| y_{j+k+1} \cdot \left(\frac{d}{dt}\right)^\nu A_{j,\lambda}(t)\right|} \leqslant R_+^{-1},$$

$$\limsup_{j\to-\infty} \sqrt[|j|]{\left| y_{j-k+1} \cdot \left(\frac{d}{dt}\right)^\nu B_{j,\lambda}(t)\right|} \leqslant R_-^{-1},$$

$\nu = 0, 1, ..., m-1$, $k \in \mathbb{Z}$. Using the relations

$$\left(\frac{d}{dt}\right)^\nu q_{j,\lambda}^{\infty}(1) = \left(\frac{d}{dt}\right)^\nu q_{j+1,\lambda}^{\infty}(0),$$

$\nu = 0, 1, ..., m-1$, $j \in \mathbb{Z}$, we conclude the locally uniform convergence of the s_ν, $\nu = 0, 1, ..., m-1$, and hence $s \in C^{m-1}(\mathbb{R})$. From (16) we obtain the identity

$$s(k-1+t)=y_k(1-t)^m\lambda^{-m}+\sum_{\mu\in I_{2,\lambda}}^{*} a_{\mu,\lambda}(t)\cdot\sum_{j=0}^{\infty} y_{k+j+1}z_{\mu,\lambda}^{j}$$

$$-\sum_{\mu\in I_{1,\lambda}}^{*} a_{\mu,\lambda}(t)\cdot\sum_{j=1}^{\infty} y_{k-j+1}z_{\mu,\lambda}^{-j} \qquad (22)$$

for $t\in[0,1]$, $k\in\mathbb{Z}$, which proves that s is an element of S_m and thus a solution of CIP(y, S_m).

Remark. Condition (20) of Theorem 1 means that the data $(y_j)_{j\in\mathbb{Z}}$ satisfy the growing condition

$$y_j=\mathcal{O}(\beta_1^j), \qquad j\to\infty,$$
$$y_j=\mathcal{O}(\beta_2^{|j|}), \qquad j\to-\infty,$$

with

$$\beta_1<|\zeta_2^{(m)}(1-\lambda)|^{-1} \qquad \text{and} \qquad \beta_2<|\zeta_1^{(m)}(1-\lambda)|,$$

i.e., $y\in Y^{(\beta_1,\beta_2)}$.

If assumption (2) is valid, then the interpolation problem CIP(y, S_m^N) has a solution s with

$$s(x)=\sum_{j=-\infty}^{\infty} y_j l_{j,\lambda}^{\infty}(x)=\sum_{j=-\infty}^{\infty} y_j l_{0,\lambda}^{\infty}(x-j)$$
$$=\sum_{j=-\infty}^{\infty} y_{j-N} l_{0,\lambda}^{\infty}(x-j+N)=s(x+N),$$

provided that $y\in Y^N$ (compare Schoenberg [19]). Moreover, if we define the data $y=(y_j)_{j\in\mathbb{Z}}$ by

$$y_j:=\begin{cases}0 & \text{if } j\not\equiv 0 \bmod N\\ 1 & \text{if } j\equiv 0 \bmod N,\end{cases}$$

then we obtain the relation

$$s(k-1+t)=(1-t)^m\lambda^{-m}\,\delta_{0,k}+\sum_{\mu\in I_{2,\lambda}} a_{\mu,\lambda}(t)\cdot\sum_{j=1}^{\infty} z_{\mu,\lambda}^{jN-k-1}$$
$$-\sum_{\mu\in I_{1,\lambda}} a_{\mu,\lambda}(t)\cdot\sum_{j=0}^{\infty} z_{\mu,\lambda}^{-jN-k-1}$$
$$=q_{k,\lambda}^N(t),$$

$t\in[0,1]$, $k=0,1,\ldots,N-1$. Thus under assumption (2) the identity (13) may be derived from the representation of $l_{0,\lambda}^{\infty}$, given in Lemma 1.

3. Uniqueness

Due to Schoenberg [14, 16] the dimension of the linear space

$$W_\lambda := \{s \in S_m : s(j-\lambda) = 0, j \in \mathbb{Z}\}$$

is given by

$$d_\lambda := \begin{cases} m-1 & \text{if } \lambda = 1 \\ m & \text{if } \lambda \in (0, 1). \end{cases}$$

To construct a basis for W_λ Schoenberg used the cardinal exponential splines, defined by

$$s(x; z) := \sum_{j=-\infty}^{\infty} z^j B_{0,m}(x-j), \qquad x, z \in \mathbb{R}.$$

By means of the relations (8) and (11) it is easy to prove the identity

$$s(k-1+t; z) = \frac{z^{k-1}}{m!} H_m(t, z^{-1}) = \frac{z^{k-1-m}}{m!} H_m(1-t, z), \tag{23}$$

$t \in [0, 1]$, $z \in \mathbb{R}\backslash\{0\}$, $k \in \mathbb{Z}$. From (23) we conclude that the functions $s_{\nu,\lambda} \in S_m$, $\nu = 1, ..., d_\lambda$,

$$s_{\nu,\lambda}(x) := s(x; z_\nu^{(m)}(\lambda)) = s(x; z_{\nu,1-\lambda})$$

are solutions of CIP(0, S_m) and that every non-trivial linear combination $\Phi(x) = \sum_{\nu=1}^{d_\lambda} \xi_\nu s_{\nu,\lambda}(x)$ satisfies

$$\lim_{k \to \infty} \frac{|\Phi(k-1+t)|}{|\zeta_1^{(m)}(\lambda)|^k} > 0 \qquad \text{or} \qquad \lim_{k \to -\infty} \frac{|\Phi(k-1+t)|}{|\zeta_2^{(m)}(\lambda)|^k} > 0, \tag{24}$$

$t \in [0, 1)\backslash\{1-\lambda\}$. As a consequence of (24)

$$W_\lambda = \text{span}\{s_{\nu,\lambda} : \nu = 1, ..., d_\lambda\},$$

(compare Schoenberg [14, 16]) and hence the cardinal interpolation problem CIP(0, $S_m^{(\beta_1,\beta_2)}$) has a unique solution, if

$$\beta_1 < |\zeta_1^{(m)}(\lambda)| = |\zeta_2^{(m)}(1-\lambda)|^{-1}$$

and (25)

$$\beta_2 < |\zeta_2^{(m)}(\lambda)|^{-1} = |\zeta_1^{(m)}(1-\lambda)|.$$

In this case the series (19) is a solution of CIP(y, $S_m^{(\beta_1,\beta_2)}$), if $y \in Y^{(\beta_1,\beta_2)}$, as is shown in the following theorem.

THEOREM 2. *Let* $m \in \mathbb{N} \backslash \{1\}$, $\lambda \in (0, 1]$, *and* $\beta_1, \beta_2 \in \mathbb{R}^+$ *with*

$$|\zeta_1^{(m)}(1-\lambda)|^{-1} < \beta_1 < |\zeta_2^{(m)}(1-\lambda)|^{-1} \tag{26}$$

and

$$|\zeta_2^{(m)}(1-\lambda)| < \beta_2 < |\zeta_1^{(m)}(1-\lambda)|. \tag{27}$$

Further let

$$(\lambda = 1 \Rightarrow m \equiv 1 \bmod 2) \qquad \text{and} \qquad (\lambda = \tfrac{1}{2} \Rightarrow m \equiv 0 \bmod 2). \tag{28}$$

Then $\mathrm{CIP}(y, S_m^{(\beta_1,\beta_2)})$ *has a unique solution* s *for any data* $y \in Y^{(\beta_1,\beta_2)}$, *which is given by the series* (19).

Proof. Let s_1, s_2 be two different solutions of $\mathrm{CIP}(y, S_m^{(\beta_1,\beta_2)})$. Then $s_1 - s_2$ is a non-trivial solution of $\mathrm{CIP}(0, S_m^{(\beta_1,\beta_2)})$, which is a contradiction to (25). Hence uniqueness of the theorem is shown, and in view of Theorem 1 we only have to prove that the growth conditions

$$\begin{aligned} s(x) &= \mathcal{O}(\beta_1^x), \qquad x \to \infty, \\ s(x) &= \mathcal{O}(\beta_2^{|x|}), \qquad x \to -\infty, \end{aligned} \tag{29}$$

are satisfied by the series (19).

From Eq. (22) we obtain for fixed $k \in \mathbb{N}$ and $t \in [0, 1]$ the inequalities

$$\begin{aligned} |s(k-1+t)| \leqslant C \cdot \Bigg\{ &\beta_1^k \lambda^{-m} + \sum_{\mu \in I_{2,\lambda}}^{*} |a_{\mu,\lambda}(t)| \cdot \sum_{j=0}^{\infty} \beta_1^{k+j+1} |z_{\mu,\lambda}|^j \\ &+ \sum_{\mu \in I_{1,\lambda}}^{*} |a_{\mu,\lambda}(t)| \sum_{j=1}^{k} \beta_1^{k-j+1} |z_{\mu,\lambda}|^{-j} \\ &+ \sum_{\mu \in I_{1,\lambda}}^{*} |a_{\mu,\lambda}(t)| \cdot \sum_{j=k+1}^{\infty} \beta_2^{|k-j+1|} |z_{\mu,\lambda}|^{-j} \Bigg\} \\ \leqslant C \cdot \beta_1^k \Bigg\{ &\lambda^{-m} + \sum_{\mu \in I_{2,\lambda}}^{*} |a_{\mu,\lambda}(t)|\, \beta_1 \cdot \sum_{j=0}^{\infty} (\beta_1 |z_{\mu,\lambda}|)^j \\ &+ \sum_{\mu \in I_{1,\lambda}}^{*} |a_{\mu,\lambda}(t)|\, \beta_1 \cdot \sum_{j=0}^{\infty} (\beta_1 |z_{\mu,\lambda}|)^{-j} \Bigg\} \\ &+ C \cdot \sum_{\mu \in I_{1,\lambda}}^{*} |a_{\mu,\lambda}(t)|\, |z_{\mu,\lambda}|^{-k-1} \cdot \sum_{j=0}^{\infty} (\beta_2 |z_{\mu,\lambda}|^{-1})^j, \end{aligned}$$

where the involved series converge by (26) and (27). Thus we have for $x>0$, $t:=x-[x]$, $k:=[x]+1$,

$$\begin{aligned}|s(x)| = |s(k-1+t)| &\leqslant C_0(\beta_1^k + |\zeta_{1,\lambda}|^{-k}) \\ &\leqslant 2C_0\beta_1^k = 2C_0\beta_1^{k-1+t}\beta_1^{1-t} \\ &\leqslant C_1\beta_1^{k-1+t} = C_1\beta_1^x,\end{aligned}$$

which shows the first part of (29). Since the second part is proved with the same methods (see Siepmann [17]), it is omitted here.

Remark. Theorem 2 is sharp in the following sense. If the assumptions of Theorem 2 are valid with the exception of (26), which is replaced by the condition

$$\beta_1 \leqslant |\zeta_1^{(m)}(1-\lambda)|^{-1},$$

then we cannot conclude the existence of a solution of CIP(y, S_m) with

$$s \in S_m^{(\beta_1,\beta_2)}. \tag{30}$$

In case of $\beta_1 < |\zeta_1^{(m)}(1-\lambda)|^{-1}$ (30) is violated by every cardinal spline, interpolating the data $y=(\delta_{0,j})_{j\in\mathbb{Z}}$, which is an immediate consequence of Lemma 1 and (24).

Now let $\beta_1 = |\zeta_1^{(m)}(1-\lambda)|^{-1}$ and $(y_j)_{j\in\mathbb{Z}}$ be a sequence of data with

$$y_j = \mathcal{O}(\beta_2^{|j|}), \qquad j\to -\infty,$$

and

$$y_j = (\zeta_1^{(m)}(1-\lambda))^{-j}, \qquad j\in\mathbb{N}.$$

Following the proof of Theorem 2 we obtain the inequality

$$\frac{|s(k-1+t)|}{|\zeta_1^{(m)}(1-\lambda)|^{-k}} \geqslant C\cdot k, \qquad k\in\mathbb{N},$$

for $t\in[0,1)\backslash\{1-\lambda\}$, where s denotes the Lagrangian spline series (19). Using (24) again, we conclude that there is no solution of CIP$(y, S_m^{(\beta_1,\beta_2)})$ for the sequence $y=(y_j)_{j\in\mathbb{Z}}$, given above. The same is true, if (27) is replaced by the condition

$$\beta_2 \leqslant |\zeta_2^{(m)}(1-\lambda)|.$$

Moreover, CIP$(y, S_m^{(\beta_1,\beta_2)})$ has no unique solution, if and only if

$$\beta_1^{-1} \leqslant |z_\nu(1-\lambda)| \leqslant \beta_2$$

holds for any $\nu\in\{1, ..., m\}$.

If the data $y=(y_j)_{j\in\mathbb{Z}}$ satisfy the condition

$$|y_{j_\nu}| \geq C\cdot|\zeta_2^{(m)}(1-\lambda)|^{-j_\nu}, \qquad \nu\in\mathbb{N},$$

or

$$|y_{-j_\nu}| \geq C\cdot|\zeta_1^{(m)}(1-\lambda)|^{j_\nu}, \qquad \nu\in\mathbb{N}_0,$$

for a strictly monotonic increasing sequence $(j_\nu)_{\nu\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}$, then the series (19) doesnot converge for $t\in[0,1)\backslash\{1-\lambda\}$. However, notice that there are data of exponential growth which have a solution of CIP(y, S_m) of the same exponential growth, although the series (19) doesnot converge, compare Schempp [13], Schoenberg [15].

In the case $m=1$ we have analogous results in Theorem 1 and Theorem 2, with slightly modified growing conditions, due to the fact that $H_1(t,\cdot)$ has at most one zero, $t\in[0,1]$. The reader is referred to Siepmann [17].

As a consequence of Theorem 2 we have

Corollary 1. *Let $m\in\mathbb{N}\backslash\{1\}$, $\lambda\in[0,1)$, and $\beta_1,\beta_2\in\mathbb{R}^+$ satisfy the conditions* (26) *and* (27). *Moreover, let* (28) *hold and define $(L_{j,\lambda})_{j\in\mathbb{N}}$ by*

$$\begin{aligned} L_{2j,\lambda} &:= l^{\infty}_{j,\lambda}, \\ L_{2j-1,\lambda} &:= l^{\infty}_{-j+1,\lambda}, \end{aligned} \tag{31}$$

$j\in\mathbb{N}$:

(a) *If the space $S_m^{(\beta_1,\beta_2)}$ is provided with the topology of locally uniform convergence, then $(L_{j,\lambda})_{j\in\mathbb{N}}$ is a Schauder-basis of $S_m^{(\beta_1,\beta_2)}$ (in the sense of Singer* [19]).

(b) *If S_m^{∞} is provided with the topology of uniform convergence, then $(L_{j,\lambda})_{j\in\mathbb{N}}$ is a Schauder-basis of S_m^{∞}.*

Proof. By Eq. (22) we conclude the uniform convergence of the series (19), if $y\in Y^{\infty}$. Thus, by Theorem 2, $(L_{j,\lambda})_{j\in\mathbb{N}}$ is a topological basis of both the spaces $S_m^{(\beta_1,\beta_2)}$ and S_m^{∞}. We have to show that the associated coefficient functionals

$$\Phi_k^{(\beta_1,\beta_2)}: S_m^{(\beta_1,\beta_2)} \to \mathbb{R}, \qquad \Phi_k^{\infty}: S_m^{\infty}\to\mathbb{R},$$

$$s=\sum_{j=1}^{\infty}\xi_j L_{j,\lambda}\to\xi_k,$$

$k\in\mathbb{N}$, are continuous, which is a direct consequence of the inequalities

$$\begin{aligned} |\Phi_k^{(\beta_1,\beta_2)}(s)| &\leq \sup\{|s(x)|: x\in[-k,k]\}, \qquad s\in S_m^{(\beta_1,\beta_2)}, \\ |\Phi_k^{\infty}(s)| &\leq \sup\{|s(x)|: x\in\mathbb{R}\}, \qquad s\in S_m^{\infty}. \end{aligned}$$

Finally, we conclude that the periodic interpolation problem CIP(y, S_m^N) has a unique solution, if $y \in Y^N$, provided that (12) holds. This is a direct consequence of Theorem 2, if

$$\begin{array}{c}(N \equiv 1 \bmod 2 \wedge \lambda = 1 \wedge m \equiv 0 \bmod 2) \\ \text{or} \\ (N \equiv 1 \bmod 2 \wedge \lambda = \tfrac{1}{2} \wedge m \equiv 1 \bmod 2)\end{array} \tag{32}$$

is not satisfied. Now let (32) hold; then every solution of CIP(0, S_m^∞) is of the form

$$s(x) = C \cdot \bar{E}_m(x) = C \cdot (-1)^j \cdot H_m(x - [x], -1),$$

$x \in [j, j+1]$, $j \in \mathbb{Z}$, $C \in \mathbb{R}$, where $\bar{E}_m$ is the so-called Euler-spline, which, by definition, is a function of period 2. Thus every solution of CIP(0, S_m^N) must possess the odd period N and the period 2, which is only possible if $C = 0$. Hence the homogeneous problem CIP(0, S_m^N) has a uniqe solution, implying that CIP(y, S_m^N) has a unique solution for any data $y \in Y^N$ even in case of (32).

4. Extremal Topological Bases of Normed Spaces

By a result of Reimer [8] and Siepmann [17] the Lagrangians $l_{j,\lambda}^N$, $j \in \mathbb{Z}$, $N \in \mathbb{N} \cup \{\infty\}$, satisfy the equalities

$$\|l_{j,\lambda}^N\| = \sup\{|l_{j,\lambda}^N(x)| : x \in \mathbb{R}\} = 1, \qquad j \in \mathbb{Z},$$

if $\lambda \in \{\frac{1}{2}, 1\}$, where we assume that (12) holds in the periodic case. Algebraic bases of normed spaces with this property belong to the class of extremal algebraic bases of a normed space X, thus allowing a stable representation of the elements of X (see Reimer [7]). We extend this concept of extremal bases to topological bases of normed spaces with infinite dimension.

Definition. Let $(X, \|\cdot\|)$ be a normed space with infinite dimension, $(x_j)_{j \in \mathbb{N}}$ be a topological basis of X and $\Phi_k : X \to \mathbb{R}$, $s = \sum_{j=1}^{\infty} \xi_j x_j \to \Phi_k(s) = \xi_k$, be the associated coefficient functionals, $k \in \mathbb{N}$. Further, let $V_k(X) \subset X$ be defined by

$$V_k(X) := \{x \in X : \Phi_k(x) = 0\},$$

$k \in \mathbb{N}$. Then $(x_j)_{j \in \mathbb{N}}$ is called an extremal topological basis of X, if $0 \in V_k(X)$ is a solution of every of the following problems of best approximation:

$$\|x_k - v^*\| = \inf\{\|x_k - v\| : v \in V_k(X)\},$$

$k \in \mathbb{N}$.

Now, let $X := S_m^\infty$ be provided with the sup-norm on $\mathbb{R}$, define $x_j := (L_{j,\lambda})_{j \in \mathbb{N}}$ by (31) and assume for the rest of the paper that (3) is valid. Since

$$\begin{aligned}\|L_{2k-1,\lambda} - s\| &\geqslant |(l^\infty_{-k,\lambda} - s)(-k-\lambda)| = 1 \\ &= \|l^\infty_{-k,\lambda}\| = \|L_{2k-1,\lambda}\|\end{aligned}$$

holds for any $s \in V_{2k-1}(S_m^\infty)$ and

$$\begin{aligned}\|L_{2k,\lambda} - s\| &\geqslant |(l^\infty_{k,\lambda} - s)(k-\lambda)| = 1 \\ &= \|l^\infty_{k,\lambda}\| = \|L_{2k,\lambda}\|\end{aligned}$$

holds for any $s \in V_{2k}(S_m^\infty)$, we recognize that $(L_{j,\lambda})_{j \in \mathbb{N}}$ is an extremal topological basis of S_m^∞.

Let the maps $\Phi^\infty : S_m^\infty \to l^\infty = \{\xi \in \mathbb{R}^{\mathbb{N}} : \xi \text{ bounded}\}$ and $\Phi^N : S_m^N \to \mathbb{R}^N$ be defined by

$$s = \sum_{j=1}^{\infty} \xi_j L_{j,\lambda} \to (\xi_j)_{j \in \mathbb{N}} =: \Phi^\infty(s), \tag{33}$$

$$s = \sum_{j=1}^{N} \xi_j l^N_{j,\lambda} \to (\xi_1, ..., \xi_N) =: \Phi^N(s), \tag{34}$$

$N \in \mathbb{N}$. It is convenient to introduce the numbers

$$\operatorname{cond} \Phi^N := |||\Phi^N||| \cdot |||(\Phi^N)^{-1}|||,$$

$N \in \mathbb{N} \cup \{\infty\}$, which are called conditions of the representations (33), (34), respectively. Here $|||\cdot|||$ is the usual operator-norm, where the spaces l^∞, $\mathbb{R}^N$ are provided with the sup-norm, $N \in \mathbb{N}$. As a consequence of the fact that $(L_{j,\lambda})_{j \in \mathbb{N}}$ and $\{l^N_{j,\lambda} : j = 1, ..., N\}$ are extremal bases, we have

$$|\xi_k| = \|\xi_k L_{k,\lambda}\| \leqslant \left\| \sum_{j=1}^{\infty} \xi_j L_{j,\lambda} \right\|, \qquad k \in \mathbb{N},$$

and

$$|\xi_k| = \|\xi_k l^N_{k,\lambda}\| \leqslant \left\| \sum_{j=1}^{N} \xi_j l^N_{j,\lambda} \right\|, \qquad k = 1, ..., N,$$

and hence

$$|||\Phi^N||| = 1,$$

$N \in \mathbb{N} \cup \{\infty\}$. Moreover, if $\|\xi\| \leqslant 1$ we obtain the inequalities

$$\begin{aligned}\|(\Phi^{\infty})^{-1}(\xi)\| &= \left\| \sum_{j=1}^{\infty} \xi_j L_{j,\lambda} \right\| \\ &= \sup\left\{ \left| \sum_{j=1}^{\infty} \xi_j L_{j,\lambda}(x) \right| : x \in \mathbb{R} \right\} \\ &\leqslant \sup\left\{ \sum_{j=1}^{\infty} |L_{j,\lambda}(x)| : x \in \mathbb{R} \right\} \\ &= \|\mathscr{L}_{m,\lambda}^{\infty}\|\end{aligned}$$

and

$$\begin{aligned}\|(\Phi^{N})^{-1}(\xi)\| &= \left\| \sum_{j=1}^{N} \xi_j l_{j,\lambda}^N \right\| \\ &= \sup\left\{ \left| \sum_{j=1}^{N} \xi_j l_{j,\lambda}^N(x) \right| : x \in \mathbb{R} \right\} \\ &\leqslant \sup\left\{ \sum_{j=1}^{N} |l_{j,\lambda}^N(x)| : x \in \mathbb{R} \right\} \\ &= \|\mathscr{L}_{m,\lambda}^{N}\|,\end{aligned}$$

where $\mathscr{L}_{m,\lambda}^{\infty}$ ($\mathscr{L}_{m,\lambda}^{N}$) denotes the cardinal ($N$-periodic) spline-interpolation operator, which maps any bounded (N-periodic) function $f: \mathbb{R} \to \mathbb{R}$ onto the unique solution of $\mathrm{CIP}((f(j-\lambda))_{j\in\mathbb{Z}}, S_m^{\infty})$ ($\mathrm{CIP}((f(j-\lambda))_{j\in\mathbb{Z}}, S_m^{N})$). Thus we have

$$\operatorname{cond} \Phi^N \leqslant |||\mathscr{L}_{m,\lambda}^{N}|||, \qquad N \in \mathbb{N} \cup \{\infty\}.$$

Note that, due to Reimer [7], the condition of an extremal basis of any finite-dimensional normed space is bounded by its dimension. Since Richards [11] proved the relation

$$|||\mathscr{L}_{m,\lambda}^{\infty}||| = \sup\{ |||\mathscr{L}_{m,\lambda}^{N}||| : N \in \mathbb{N} \} = \lim_{N\to\infty} |||\mathscr{L}_{m,\lambda}^{N}|||,$$

the bound given above is small, compared with the dimension of S_m^N in the periodic case $N \in \mathbb{N}$, if N is chosen sufficiently high. For the calculation of $|||\mathscr{L}_{m,\lambda}^{N}|||$, $N \in \mathbb{N} \cup \{\infty\}$, and the asymptotic behaviour of $|||\mathscr{L}_{m,\lambda}^{\infty}|||$, $m \to \infty$, see Meinardus [2], Meinardus and Merz [3], Merz [4], Richards [11, 12].

Finally let us remark that we conjecture that the sums of the squares of the Lagrangians, involved in (33) and (34), are bounded by one, if (3) is valid. This has been shown in the case of $m = 3$, $\lambda = 1$ by Siepmann and

Sündermann [18] and may be veryfied for $m=2$, $\lambda=\frac{1}{2}$, using the same method, and in the case of $m=1$, $\lambda=1$ by direct calculation.

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